

# Tsallis entropy and general polygamy of multi-party quantum entanglement in arbitrary dimensions

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We establish a unified view to the polygamy of multi-party quantum entanglement in arbitrary dimensions. Using quantum Tsallis- $q$  entropy, we provide a one-parameter class of polygamy inequalities of multi-party quantum entanglement. This class of polygamy inequalities reduces to the known polygamy inequalities based on tangle and entanglement of assistance for a selective choice of the parameter  $q$ . We further provide one-parameter generalizations of various quantum correlations based on Tsallis- $q$  entropy. By investigating the properties of the generalized quantum correlations, we provide a sufficient condition, on which the Tsallis- $q$  polygamy inequalities hold in multi-party quantum systems of arbitrary dimensions.

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## I. INTRODUCTION

Quantum entanglement is a quintessential manifestation of quantum mechanics revealing the fundamental insights into the nature of quantum correlations. One distinct property of quantum entanglement from other classical correlations is its limited shareability in multi-party quantum systems, known as the *monogamy of entanglement* (MoE) [1, 2].

MoE was characterized in a quantitative way as an inequality; for a given three-party quantum state  $\rho_{ABC}$  with reduced density matrices  $\rho_{AB} = \text{tr}_C \rho_{ABC}$  and  $\rho_{AC} = \text{tr}_B \rho_{ABC}$ , and a bipartite entanglement measure  $E$ , *monogamy inequality* leads

$$E(\rho_{A|BC}) \geq E(\rho_{A|B}) + E(\rho_{A|C}) \quad (1)$$

where  $E(\rho_{A|BC})$  is the bipartite entanglement between subsystems  $A$  and  $BC$ . Monogamy inequality shows the mutually exclusive relation of the bipartite entanglement between  $A$  and each of  $B$  and  $C$  (measured by  $E(\rho_{A|B})$  and  $E(\rho_{A|C})$ , respectively), so that their summation cannot exceed the total entanglement between  $A$  and  $BC$  (measured by  $E(\rho_{A|BC})$ ).

Monogamy inequality was first proven for three-qubit systems using *tangle* as the bipartite entanglement measure [3], and generalized into multi-qubit systems in terms of various entanglement measures [4–7]. For a general monogamy inequality of multi-party quantum entanglement in arbitrary dimension, it was shown that squashed entanglement [8] is a faithful entanglement measure [9], which also shows a general monogamy inequality [10].

Whereas MoE is about the limited shareability of bipartite entanglement in multi-party quantum systems,

the *assisted entanglement*, which is a dual amount to bipartite entanglement measures, is known to have a dually monogamous (thus polygamous) property in multi-party quantum systems. Moreover, this dually monogamous property of multi-party quantum entanglement was also characterized as a dual monogamy inequality (thus polygamy inequality) [11],

$$\tau_a(\rho_{A|BC}) \leq \tau_a(\rho_{A|B}) + \tau_a(\rho_{A|C}), \quad (2)$$

for a three-qubit state  $\rho_{ABC}$ , where  $\tau_a(\rho_{A|BC})$  is the tangle of assistance of  $\rho_{ABC}$  with respect to the bipartition between  $A$  and  $BC$ . Later, Inequality (2) was generalized into multi-qubit systems as well as some class of higher-dimensional quantum systems [6, 12]. A general polygamy inequality of multi-party quantum entanglement in arbitrary dimensional quantum systems was established using entanglement of assistance [13, 14].

As a one-parameter generalization of von Neumann entropy, Tsallis- $q$  entropy [15, 16] is used in many areas of quantum information theory; Tsallis entropy provides some conditions for separability of quantum states [17–19], and it is used to characterize classical statistical correlations inherent in quantum states [20]. There are also discussions about using the non-extensive statistical mechanics to describe quantum entanglement in terms of Tsallis entropy [21].

Tsallis entropy also plays an important role in quantum entanglement theory. For all parameters  $q > 0$ , Tsallis- $q$  entropy is a concave function on the set of density matrices, which assures the property of *entanglement monotone* [22]. In other words, Tsallis entropy can be used to construct a faithful entanglement measure that does not increase under *local quantum operations and classical communication* (LOCC).

Here, we establish a unified view to polygamy inequalities of multi-party quantum entanglement in terms of Tsallis- $q$  entropy. Using a class of bipartite entanglement measures, *Tsallis- $q$  entanglement* as well as its dual quantities *Tsallis- $q$  entanglement of assistance*, we pro-

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vide a one-parameter class of polygamy inequalities in multi-party quantum systems of arbitrary dimensions.

This class of polygamy inequalities is reduced to the known polygamy inequalities based on tangle and entanglement of assistance for a selective choice of the parameter  $q$ . Thus our class of polygamy inequalities provides an interpolation among various polygamy inequalities of multi-party quantum entanglement.

We further provide one-parameter generalizations of various quantum correlations based on Tsallis- $q$  entropy. By investigating the properties of the generalized quantum correlations, we provide a sufficient condition, on which the Tsallis- $q$  polygamy inequality holds in multi-party quantum systems of arbitrary dimensions. Moreover, we show that the sufficient condition we provide here is guaranteed for the polygamy inequality based on entanglement of assistance. Thus our results also encapsulate the known results of general polygamy inequality in a unified view in terms of Tsallis- $q$  entropy.

This paper is organized as follows. In Sec. II, we recall the definition of Tsallis- $q$  entropy, and provide some generalize entropic properties in terms of Tsallis- $q$  entropy. In Sec. III A, we recall the definitions of Tsallis- $q$  entanglement as well as its dual quantity, Tsallis- $q$  entanglement of assistance (TEoA), and we briefly review the monogamy and polygamy inequalities in multi-party quantum systems based on generalized entropies in Sec. III B. In Sec. III C, we provide a unified view of general polygamy inequality of multi-party quantum entanglement using TEoA. In Sec. IV, we generalize various quantum correlations such as Holevo quantity, one-way unlocalizable entanglement and quantum mutual information into one-parameter classes with respect to the parameter  $q$ . In Sec. V, we consider a classical-classical-quantum state in four-party quantum systems, and investigate its properties related with the generalized quantum correlations in the previous section. In Sec. VI, we show some sufficient condition for the general polygamy inequality of multi-party quantum entanglement in arbitrary dimensions using Tsallis- $q$  entropy, and we summarize our results in Sec. VII.

## II. TSALLIS- $q$ ENTROPY

Based on the generalized logarithmic function with respect to the parameter  $q$  with  $q > 0$ ,  $q \neq 1$ ,

$$\ln_q x = \frac{x^{1-q} - 1}{1 - q}, \quad (3)$$

Tsallis- $q$  entropy (or Tsallis entropy of order  $q$ ) for a probability distribution  $\mathbf{P} = \{p_i\}$  is defined as

$$H_q(\mathbf{P}) = - \sum_i p_i^q \ln_q p_i = \frac{1}{1 - q} \left[ \sum_i p_i^q - 1 \right], \quad (4)$$

which takes the  $q$ -expectation of the generalized logarithmic function with respect to the probability distribu-

tion [15]. As the singularity at  $q = 1$  in Eq. (3) is removable by its limit value, which is the natural logarithm  $\ln x$ , Tsallis- $q$  entropy in Eq. (4) converges to Shannon entropy when  $q$  tends to 1,

$$\lim_{q \rightarrow 1} H_q(\mathbf{P}) = - \sum_i p_i \ln p_i = H(\mathbf{P}). \quad (5)$$

By replacing the probability distribution  $\mathbf{P}$  with a density matrix  $\rho$ , quantum Tsallis- $q$  entropy is defined as

$$S_q(\rho) = -\text{tr} \rho^q \ln_q \rho = \frac{1 - \text{tr}(\rho^q)}{q - 1} \quad (6)$$

for  $q > 0$ ,  $q \neq 1$  [16]. Similarly, quantum Tsallis- $q$  entropy converges to von Neumann entropy when  $q$  tends to 1,

$$\lim_{q \rightarrow 1} S_q(\rho) = -\text{tr} \rho \ln \rho = S(\rho). \quad (7)$$

For these reasons, we simply denote  $S_1(\rho) = S(\rho)$ , and thus Tsallis- $q$  entropy is a one-parameter generalization of von Neumann entropy with respect to the parameter  $q$ .

It is noteworthy that Tsallis- $q$  entropy is a nonextensive generalization of von Neumann entropy. Whereas von Neumann entropy has the *extensivity* (or additivity) property, that is, the joint entropy of a pair of independent systems  $\rho \otimes \sigma$  is equal to the sum of the individual entropies

$$S(\rho \otimes \sigma) = S(\rho) + S(\sigma), \quad (8)$$

this extensivity no longer holds for Tsallis- $q$  entropy, unless  $q = 1$ . Instead, Tsallis- $q$  entropy has so-called *pseudoaddivitivity* relation as

$$S_q(\rho \otimes \sigma) = S_q(\rho) + S_q(\sigma) + (1 - q) S_q(\rho) S_q(\sigma) \quad (9)$$

for  $q \geq 0$ .

The following lemma shows that the idea of  $q$ -expectation naturally generalizes some entropic property in terms of Tsallis- $q$  entropy.

**Lemma 1.** (*Joint entropy theorem*) For a probability distribution  $\mathbf{P} = \{p_i\}$ , a set of density operators  $\{\rho_A^i\}$  of a system  $A$  and a set of orthogonal states  $\{|i\rangle_B\}$  of another system  $B$ , we have

$$S_q \left( \sum_i p_i \rho_A^i \otimes |i\rangle_B \langle i| \right) = \sum_i p_i^q S_q(\rho_A^i) + H_q(\mathbf{P}), \quad (10)$$

for  $q \geq 0$  and  $q \neq 1$ .

*Proof.* From the definition of quantum Tsallis- $q$  entropy

in Eq. (6),

$$\begin{aligned}
S_q \left( \sum_i p_i \rho_A^i \otimes |i\rangle_B \langle i| \right) &= \frac{1 - \text{tr} \left( \sum_i p_i \rho_A^i \otimes |i\rangle_B \langle i| \right)^q}{q-1} \\
&= \frac{1 - \text{tr} \sum_i p_i^q (\rho_A^i)^q}{q-1} \\
&= \frac{1 - \sum_i p_i^q}{q-1} + \sum_i p_i^q \frac{1 - \text{tr} (\rho_A^i)^q}{q-1} \\
&= H_q(\mathbf{P}) + \sum_i p_i^q S_q(\rho_A^i), \tag{11}
\end{aligned}$$

□

In fact, we can analogously show that Eq. (10) also holds in more general cases; for a probability distribution  $\mathbf{P} = \{p_i\}$  and a set of density operators  $\{\rho^i\}$  with mutually orthogonal supports, we have

$$S_q \left( \sum_i p_i \rho^i \right) = \sum_i p_i^q S_q(\rho^i) + H_q(\mathbf{P}), \tag{12}$$

for  $q \geq 0$  and  $q \neq 1$ .

Due to the continuity of Tsallis- $q$  entropy with respect to the parameter  $q$ , Eq. (10) is reduced to the joint entropy theorem in terms of Shannon and von Neumann entropy,

$$S \left( \sum_i p_i \rho_A^i \otimes |i\rangle_B \langle i| \right) = \sum_i p_i S(\rho_A^i) + H(\mathbf{P}), \tag{13}$$

for the case when  $q$  tends to 1.

### III. TSALLIS ENTANGLEMENT AND POLYGAMY OF MULTI-PARTY QUANTUM ENTANGLEMENT

#### A. Tsallis- $q$ entanglement

For a bipartite pure state  $|\psi\rangle_{AB}$  with its reduced density matrix  $\rho_A = \text{tr}_B |\psi\rangle_{AB} \langle \psi|$  onto subsystem  $A$ , its Tsallis- $q$  entanglement is defined as [6]

$$\mathcal{T}_q(|\psi\rangle_{AB}) = S_q(\rho_A). \tag{14}$$

For a bipartite mixed state  $\rho_{AB}$ , its Tsallis- $q$  entanglement is defined via convex-roof extension,

$$\mathcal{T}_q(\rho_{AB}) = \min \sum_i p_i \mathcal{T}_q(|\psi_i\rangle_{AB}), \tag{15}$$

where the minimization is taken over all possible pure state decompositions of  $\rho_{AB}$ ,

$$\rho_{AB} = \sum_i p_i |\psi^i\rangle_{AB} \langle \psi^i|. \tag{16}$$

Because Tsallis- $q$  entropy converges to von Neumann entropy when  $q$  tends to 1, we have

$$\lim_{q \rightarrow 1} \mathcal{T}_q(\rho_{AB}) = E_f(\rho_{AB}), \tag{17}$$

where  $E_f(\rho_{AB})$  is the *entanglement of formation*(EoF) [23] of  $\rho_{AB}$ , defined as

$$\begin{aligned}
E_f(\rho_{AB}) &= \min \sum_i p_i S(\rho_A^i) \\
&= \min \sum_i p_i S(\rho_B^i) \\
&= E_f(\rho_{B|A}) = E_f(\rho_{AB}). \tag{18}
\end{aligned}$$

with the minimum taken over all possible pure state decompositions of  $\rho_{AB}$  in Eq. (16),  $\rho_A^i = \text{tr}_B |\psi^i\rangle_{AB} \langle \psi^i|$  and  $\rho_B^i = \text{tr}_A |\psi^i\rangle_{AB} \langle \psi^i|$ . Moreover, due to the coincidence

$$S_q(\rho_A^i) = S_q(\rho_B^i) \tag{19}$$

for each  $|\psi^i\rangle_{AB}$  in Eq. (16), we have

$$\begin{aligned}
\mathcal{T}_q(\rho_{AB}) &= \min \sum_i p_i S_q(\rho_A^i) \\
&= \min \sum_i p_i S_q(\rho_B^i) \\
&= \mathcal{T}_q(\rho_{B|A}). \tag{20}
\end{aligned}$$

As a dual quantity to Tsallis- $q$  entanglement, Tsallis- $q$  entanglement of Assistance(TEoA) is defined as [6]

$$\mathcal{T}_q^a(\rho_{AB}) = \max \sum_i p_i \mathcal{T}_q(|\psi_i\rangle_{AB}), \tag{21}$$

where the maximum is taken over all possible pure state decompositions of  $\rho_{AB}$ . Similarly, we have

$$\lim_{q \rightarrow 1} \mathcal{T}_q^a(\rho_{AB}) = E^a(\rho_{AB}), \tag{22}$$

where  $E^a(\rho_{AB})$  is the *entanglement of assistance*(EoA) of  $\rho_{AB}$  defined as [24]

$$E^a(\rho_{AB}) = \max \sum_i p_i S(\rho_A^i). \tag{23}$$

with the maximization over all possible pure state decompositions of  $\rho_{AB}$ .

#### B. Monogamy and polygamy inequalities of multi-party quantum entanglement based on generalized quantum entropies

Using Tsallis- $q$  entanglement in Eq. (15) to quantify bipartite quantum entanglement, the monogamy inequality in Eq. (1) was established in multi-qubit systems; for any  $n$ -qubit state  $\rho_{A_1 A_2 \dots A_n}$  and its two-qubit reduced density matrices  $\rho_{A_1 A_i}$  with  $i = 2, \dots, n$ , we have

$$\mathcal{T}_q(\rho_{A_1 | A_2 \dots A_n}) \geq \mathcal{T}_q(\rho_{A_1 | A_2}) + \dots + \mathcal{T}_q(\rho_{A_1 | A_n}), \tag{24}$$

for  $2 \leq q \leq 3$  [6]. It was also shown that TEOA can be used to characterize the polygamy of multi-qubit entanglement as

$$\mathcal{T}_q^a(\rho_{A_1|A_2 \dots A_n}) \leq \mathcal{T}_q^a(\rho_{A_1|A_2}) + \dots + \mathcal{T}_q^a(\rho_{A_1|A_n}), \quad (25)$$

for  $1 \leq q \leq 2$  and  $3 \leq q \leq 4$  [6]. Recently, more generalized monogamy and polygamy inequalities of multi-qubit entanglement was proposed in terms of Tsallis- $q$  entanglement and TEOA for selective choices of  $q$  [25].

Besides Tsallis- $q$  entropy, Rényi- $\alpha$  entropy is another one-parameter family of entropy functions, which contains von Neumann entropy as a special case; for a positive real number  $\alpha$  and a quantum state  $\rho$ , the Rényi- $\alpha$  entropy of  $\rho$  is defined as

$$R_\alpha(\rho) = \frac{1}{1-\alpha} \log \text{tr} \rho^\alpha \quad (26)$$

for  $\alpha \neq 1$  [26, 27]. Similar to the case of Tsallis- $q$  entropy, Rényi- $\alpha$  entropy has a singularity at  $\alpha = 1$ . However this singularity is removable in the sense that Rényi- $\alpha$  entropy converges to von Neumann entropy when  $\alpha$  tends to 1.

As a generalization of EoF into the full spectrum of Rényi- $\alpha$  entropy, Rényi- $\alpha$  entanglement was introduced as

$$E_\alpha(|\psi\rangle_{A|B}) = R_\alpha(\rho_A), \quad (27)$$

for a bipartite pure state  $|\psi\rangle_{AB}$  and

$$E_\alpha(\rho_{A|B}) = \min_i \sum_i p_i E_\alpha(|\psi_i\rangle_{A|B}), \quad (28)$$

for a bipartite mixed state  $\rho_{AB}$  with the minimum over all possible pure-state decompositions of  $\rho_{A|B} = \sum_i p_i |\psi_i\rangle_{AB} \langle \psi_i|$  [5, 28].

Based on Rényi- $\alpha$  entanglement of order 2 (Rényi-2 entanglement), a monogamy inequality was established for multi-qubit entanglement as

$$E_2(\rho_{A_1|A_2 \dots A_n}) \geq E_2(\rho_{A_1|A_2}) + \dots + E_2(\rho_{A_1|A_n}), \quad (29)$$

for a multi-qubit state  $\rho_{A_1 A_2 \dots A_n}$  and its two-qubit reduced density matrices  $\rho_{A_1 A_i}$  [29]. Later, the validity of multi-qubit Rényi- $\alpha$  monogamy inequality was shown for any  $\alpha \geq 2$ , that is,

$$E_\alpha(\rho_{A_1|A_2 \dots A_n}) \geq E_\alpha(\rho_{A_1|A_2}) + \dots + E_\alpha(\rho_{A_1|A_n}), \quad (30)$$

for any multi-qubit state  $\rho_{A_1 A_2 \dots A_n}$  and  $\alpha \geq 2$  [5].

### C. Unification of polygamy inequalities

The first polygamy inequality was established in three-qubit systems [11]; for a three-qubit pure state  $|\psi\rangle_{ABC}$ ,

$$\tau(|\psi\rangle_{A|BC}) \leq \tau^a(\rho_{A|B}) + \tau^a(\rho_{A|C}), \quad (31)$$

where

$$\tau(|\psi\rangle_{A|BC}) = 4 \det \rho_A \quad (32)$$

is the tangle of the pure state  $|\psi\rangle_{ABC}$  between  $A$  and  $BC$ , and

$$\tau^a(\rho_{A|B}) = \max_i \sum_i p_i \tau(|\psi_i\rangle_{A|B}) \quad (33)$$

is the tangle of assistance of  $\rho_{AB} = \text{tr}_C |\psi\rangle_{ABC} \langle \psi|$  with the maximum taken over all pure-state decompositions of  $\rho_{AB}$ . Later, Inequality (31) was generalized into multi-qubit systems [12]

$$\tau^a(\rho_{A_1|A_2 \dots A_n}) \leq \tau^a(\rho_{A_1|A_2}) + \dots + \tau^a(\rho_{A_1|A_n}), \quad (34)$$

for an arbitrary multi-qubit mixed state  $\rho_{A_1 \dots A_n}$  and its two-qubit reduced density matrices  $\rho_{A_1 A_i}$  with  $i = 2, \dots, n$ .

For polygamy inequality beyond qubits, it was shown that EoA can be used to establish a polygamy inequality of three-party quantum systems as

$$E^a(|\psi\rangle_{A|BC}) \leq E^a(\rho_{A|B}) + E^a(\rho_{A|C}) \quad (35)$$

for any three-party pure state  $|\psi\rangle_{ABC}$  of arbitrary dimensions [13]. A general polygamy inequality was established by generalizing EoA polygamy inequality in (35) into multi-party quantum systems as

$$E^a(\rho_{A_1|A_2 \dots A_n}) \leq E^a(\rho_{A_1|A_2}) + \dots + E^a(\rho_{A_1|A_n}), \quad (36)$$

for any multi-party quantum state  $\rho_{A_1 A_2 \dots A_n}$  of arbitrary dimension [14].

Now, let us consider an unified view of the polygamy inequalities of multi-party entanglement in terms of Tsallis- $q$  entropy. For any two-qubit pure state  $|\psi\rangle_{AB}$  (or any bipartite state with Schmidt-rank 2) with a Schmidt decomposition

$$|\psi\rangle_{AB} = \sqrt{\lambda_1} |e_0\rangle_A \otimes |f_0\rangle_B + \sqrt{\lambda_2} |e_1\rangle_A \otimes |f_1\rangle_B, \quad (37)$$

its tangle in Eq. (32) coincides with Tsallis-2 entanglement up to a constant factor

$$\tau(|\psi\rangle_{A|B}) = 4\lambda_0\lambda_1 = 2\mathcal{T}_2(|\psi\rangle_{A|B}). \quad (38)$$

Thus the tangle-based polygamy inequality in (34) can be rephrased as

$$\mathcal{T}_2^a(\rho_{A_1|A_2 \dots A_n}) \leq \mathcal{T}_2^a(\rho_{A_1|A_2}) + \dots + \mathcal{T}_2^a(\rho_{A_1|A_n}), \quad (39)$$

for any multi-qubit state  $\rho_{A_1 \dots A_n}$ .

Due to the continuity of Tsallis- $q$  entropy, the relation between TEOA and EoA in Eq. (22) enables us to rephrase EoA-based polygamy inequality in (36) as

$$\mathcal{T}_1^a(\rho_{A_1|A_2 \dots A_n}) \leq \mathcal{T}_1^a(\rho_{A_1|A_2}) + \dots + \mathcal{T}_1^a(\rho_{A_1|A_n}). \quad (40)$$

In other words, the polygamy inequalities of multi-party quantum entanglement established so far can be considered in an unified way using Tsallis- $q$  entropy as

$$\mathcal{T}_q^a(\rho_{A_1|A_2\cdots A_n}) \leq \mathcal{T}_q^a(\rho_{A_1|A_2}) + \cdots + \mathcal{T}_q^a(\rho_{A_1|A_n}), \quad (41)$$

for selective choices of  $q$ .

In the following sections, we investigate some properties of quantum correlations based on Tsallis- $q$  entropy, and provide sufficient conditions, on which the Tsallis- $q$  polygamy inequality in (41) holds.

#### IV. $q$ -EXPECTATION AND QUANTUM CORRELATIONS

The definition of Tsallis- $q$  entropy in Eq. (6) uses the concept of  $q$ -expectation to generalize von-Neumann entropy into a class of entropies parameterized by  $q$ . Here, we further generalize some quantum correlations based on the idea of  $q$ -expectation, and investigate their properties.

For a quantum state  $\rho$  and its ensemble representation  $\mathcal{E} = \{p_i, \rho_i\}$  (equivalently, a probability decomposition  $\rho = \sum_i p_i \rho_i$ ), *Tsallis- $q$  difference* is defined as

$$\chi_q(\mathcal{E}) = S_q(\rho) - \sum_i p_i^q S_q(\rho_i), \quad (42)$$

which is a one-parameter generalization of the Holevo quantity,

$$\chi(\mathcal{E}) = S(\rho) - \sum_i p_i S(\rho_i), \quad (43)$$

for  $q = 1$ . Due to the concavity of Tsallis- $q$  entropy, Tsallis- $q$  difference is always nonnegative for  $q \geq 1$ .

Now, let us consider a bipartite quantum state  $\rho_{AB}$  with its reduced density matrix  $\rho_A = \text{tr}_B \rho_{AB}$ . Each rank-1 measurement  $\{M_x\}$  applied on subsystem  $B$  induces a probability ensemble  $\mathcal{E} = \{p_x, \rho_A^x\}$  of  $\rho_A$  where  $p_x \equiv \text{tr}[(I_A \otimes M_x)\rho_{AB}]$  is the probability of the outcome  $x$  and  $\rho_A^x \equiv \text{tr}_B[(I_A \otimes M_x)\rho_{AB}]/p_x$  is the state of system  $A$  when the outcome was  $x$ . For  $q \geq 1$ , we define *one-way unlocalizable  $q$ -entanglement* ( $q$ -UE) as the minimum Tsallis- $q$  difference

$$\mathbf{u}E_q^{\leftarrow}(\rho_{AB}) = \min_{\mathcal{E}} \chi_q(\mathcal{E}), \quad (44)$$

where the minimum is taken over the ensemble representations  $\mathcal{E} = \{p_x, \rho_A^x\}$  of  $\rho_A$  induced by all possible rank-1 measurements  $\{M_x\}$  on subsystem  $B$ .

Due to the continuity of Tsallis- $q$  entropy with respect to the parameter  $q$ ,  $q$ -UE is reduced to the one-way unlocalizable entanglement

$$\mathbf{u}E^{\leftarrow}(\rho_{AB}) = \min_{\mathcal{E}} \chi(\mathcal{E}), \quad (45)$$

when  $q$  tends to 1 [13].

The term *unlocalizable* arises for the following reasons. Eq. (44) together with Eq. (42) enable us to rewrite  $q$ -UE as

$$\mathbf{u}E_q^{\leftarrow}(\rho_{AB}) = S_q(\rho_A) - \max_{\{M_x\}} \sum_x p_x^q S_q(\rho_A^x) \quad (46)$$

where the maximum is taken over all possible rank-1 measurements  $\{M_x\}$  applied on system  $B$ .

For a three-party purification  $|\psi\rangle_{ABC}$  of  $\rho_{AB}$  such that  $\text{tr}_C |\psi\rangle_{ABC} \langle \psi| = \rho_{AB}$ , we note that each rank-1 measurement  $\{M_x\}$  applied on system  $B$  induces a pure-state decomposition of  $\rho_{AC} = \text{tr}_B |\psi\rangle_{ABC} \langle \psi|$  as

$$\rho_{AC} = \sum_x p_x |\phi^x\rangle_{AC} \langle \phi^x| \quad (47)$$

where  $p_x \equiv \text{tr}[(I_{AC} \otimes M_x)|\psi\rangle_{ABC} \langle \psi|]$  and  $|\phi^x\rangle_{AC} \langle \phi^x| \equiv \text{tr}_B[(I_{AC} \otimes M_x)|\psi\rangle_{ABC} \langle \psi|]/p_x$ . Moreover, it is also straightforward to verify that each pure-state decomposition of  $\rho_{AC} = \sum_x p_x |\phi^x\rangle_{AC} \langle \phi^x|$  induces a rank-1 measurement  $\{M_x\}$  applied on system  $B$ . Because we have

$$\text{tr}_C |\phi^x\rangle_{AC} \langle \phi^x| = \rho_A^x, \quad (48)$$

for each  $x$ , Eq. (46) can be rewritten as

$$\mathbf{u}E_q^{\leftarrow}(\rho_{AB}) = \mathcal{T}_q(|\psi\rangle_{A|BC}) - \max_{\{M_x\}} \sum_x p_x^q \mathcal{T}_q(|\phi^x\rangle_{A|C}). \quad (49)$$

Here,  $\mathcal{T}_q(|\psi\rangle_{A|BC}) = S_q(\rho_A)$  represents the amount of entanglement of the pure state  $|\psi\rangle_{ABC}$  between  $A$  and  $BC$  quantified by Tsallis- $q$  entanglement, and  $\max \sum_x p_x^q \mathcal{T}_q(|\phi^x\rangle_{A|C})$  is the maximum average entanglement (with respect to  $q$ -expectation) that is possible to be concentrated on the subsystem  $AC$  with the assistance of  $B$ . Thus  $\mathbf{u}E_q^{\leftarrow}(\rho_{AB})$  is the residual entanglement that cannot be localized (therefore unlocalizable) on  $AC$  by the local measurement of  $B$ .

From the convexity of the function  $f(x) = x^q$  for  $q \geq 1$  and the definition of TEOA in Eq. (21), we have

$$\mathcal{T}_q^a(\rho_{A|C}) \geq \max_x \sum_x p_x^q \mathcal{T}_q(|\phi^x\rangle_{A|C}), \quad (50)$$

and this leads Eq. (49) to

$$\mathbf{u}E_q^{\leftarrow}(\rho_{AB}) \geq \mathcal{T}_q(|\psi\rangle_{A|BC}) - \mathcal{T}_q(\rho_{A|C}), \quad (51)$$

for  $q \geq 1$ . Analogously, we also have

$$\mathbf{u}E_q^{\leftarrow}(\rho_{AC}) \geq \mathcal{T}_q(|\psi\rangle_{A|BC}) - \mathcal{T}_q(\rho_{A|B}). \quad (52)$$

To end this section, we provide a one-parameter generalization of quantum mutual information using Tsallis- $q$  entropy; for a bipartite quantum state  $\rho_{AB}$  with reduced

density matrices  $\rho_A = \text{tr}_B \rho_{AB}$  and  $\rho_B = \text{tr}_A \rho_{AB}$ , the *Tsallis- $q$  mutual entropy* is defined as

$$I_q(\rho_{A:B}) = S_q(\rho_A) + S_q(\rho_B) - S_q(\rho_{AB}) \quad (53)$$

for  $q \geq 1$ .

Due to the continuity of Tsallis- $q$  entropy, the Tsallis- $q$  mutual entropy in Eq. (53) is reduced to the quantum mutual information,

$$I(\rho_{A:B}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}), \quad (54)$$

for the case that  $q$  tends to 1. However, we do not use the term *mutual information* for Eq. (53) because a proper evidence of channel coding theorem for information transmission has not been shown in the context of Tsallis entropy, even in classical sense.

## V. CLASSICAL-CLASSICAL-QUANTUM(CCCQ) STATES

In this section, we consider a four-party classical-classical-quantum(ccq) state  $\Omega_{XYAB}$  whose quantum part  $AB$  is obtained from a given bipartite quantum state  $\rho_{AB}$  by applying local unitary operations depending on the classical part  $A$  and  $B$ . We also evaluate the Tsallis- $q$  mutual entropies of  $\Omega_{XYAB}$  as well as its reduced density matrices, which will provide some sufficient condition for the general polygamy inequality of multi-party quantum entanglement in terms of TEoA.

For a two-qudit quantum state  $\rho_{AB}$  in  $\mathcal{H}_A \otimes \mathcal{H}_B \simeq \mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^d)$  and the reduced density matrix  $\rho_B = \text{tr}_A(\rho_{AB})$ , let us consider a spectral decomposition,

$$\rho_B = \sum_{i=0}^{d-1} \lambda_i |e_i\rangle_B \langle e_i|. \quad (55)$$

Using the eigenvectors of  $\rho_B$ , we define two quantum channels  $M_0$  and  $M_1$

$$\begin{aligned} M_0(\sigma) &= \sum_{i=0}^{d-1} |e_i\rangle \langle e_i| \sigma |e_i\rangle \langle e_i| \\ M_1(\sigma) &= \sum_{i=0}^{d-1} |\tilde{e}_i\rangle \langle \tilde{e}_i| \sigma |\tilde{e}_i\rangle \langle \tilde{e}_i|, \end{aligned} \quad (56)$$

acting on any quantum state  $\sigma$  of subsystem  $\mathcal{H}_B$ , where  $\{|\tilde{e}_j\rangle\}_j$  is the  $d$ -dimensional *Fourier basis*,

$$|\tilde{e}_j\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \omega_d^{jk} |e_k\rangle, \quad j = 0, \dots, d-1, \quad (57)$$

and  $\omega_d = e^{\frac{2\pi i}{d}}$  is the  $d$ th-root of unity. By using the

generalized  $d$ -dimensional Pauli operators

$$\begin{aligned} Z &= \sum_{j=0}^{d-1} \omega_d^j |e_j\rangle \langle e_j|, \\ X &= \sum_{j=0}^{d-1} |e_{j+1}\rangle \langle e_j| = \sum_{j=0}^{d-1} \omega_d^{-j} |\tilde{e}_j\rangle \langle \tilde{e}_j|, \end{aligned} \quad (58)$$

Eqs. (56) can be rewritten as

$$M_0(\sigma) = \frac{1}{d} \sum_{b=0}^{d-1} Z^b \sigma Z^{-b}, \quad M_1(\sigma) = \frac{1}{d} \sum_{a=0}^{d-1} X^a \sigma X^{-a}. \quad (59)$$

The channels  $M_0$  and  $M_1$  act on  $\rho_B$  as

$$M_0(\rho_B) = \rho_B, \quad M_1(\rho_B) = \frac{1}{d} I_B, \quad (60)$$

and

$$M_1(M_0(\rho_B)) = M_0(M_1(\rho_B)) = \frac{1}{d} I_B, \quad (61)$$

thus the actions of the channels  $M_0$  and  $M_1$  on the subsystem  $B$  of the bipartite state  $\rho_{AB}$  are

$$\begin{aligned} (I_A \otimes M_0)(\rho_{AB}) &= \sum_{i=0}^{d-1} \sigma_A^i \otimes \lambda_i |e_i\rangle_B \langle e_i|, \\ (I_A \otimes M_1)(\rho_{AB}) &= \sum_{j=0}^{d-1} \tau_A^j \otimes \frac{1}{d} |\tilde{e}_j\rangle_B \langle \tilde{e}_j|, \end{aligned} \quad (62)$$

where  $\lambda_i \sigma_A^i = \text{tr}_B[(I_A \otimes |e_i\rangle_B \langle e_i|) \rho_{AB}]$  and  $\tau_A^j/d = \text{tr}_B[(I_A \otimes |\tilde{e}_j\rangle_B \langle \tilde{e}_j|) \rho_{AB}]$  for  $i, j \in \{0, \dots, d-1\}$ .

The ensembles of subsystem  $A$  induced by the action of the channels  $M_0$  and  $M_1$  on subsystem  $B$  are

$$\mathcal{E}_0 = \{\lambda_i, \sigma_A^i\}_i, \quad \mathcal{E}_1 := \left\{ \frac{1}{d}, \tau_A^j \right\}_j, \quad (63)$$

and their Tsallis- $q$  differences are

$$\chi_q(\mathcal{E}_0) = S_q(\rho_A) - \sum_{i=0}^{d-1} \lambda_i^q S_q(\sigma_A^i) \quad (64)$$

and

$$\chi_q(\mathcal{E}_1) = S_q(\rho_A) - \frac{1}{d^q} \sum_{j=0}^{d-1} S_q(\tau_A^j), \quad (65)$$

respectively.

Now, let us consider a four-qudit ccq-state  $\Omega_{XYAB}$  in  $\mathcal{H}_X \otimes \mathcal{H}_Y \otimes \mathcal{H}_A \otimes \mathcal{H}_B$ ,

$$\Omega_{XYAB} := \frac{1}{d^2} \sum_{x,y=0}^{d-1} |x\rangle_X \langle x| \otimes |y\rangle_Y \langle y| \otimes (I_A \otimes X_B^x Z_B^y) \rho_{AB} (I_A \otimes Z_B^{-y} X_B^{-x}), \quad (66)$$

with the reduced density matrices

$$\Omega_{XAB} = \frac{1}{d} \sum_{x=0}^{d-1} |x\rangle_X \langle x| \otimes X_B^x \left( \sum_{i=0}^{d-1} \sigma_A^i \otimes \lambda_i |e_i\rangle_B \langle e_i| \right) X_B^{-x}, \quad (67)$$

$$\Omega_{YAB} = \frac{1}{d} \sum_{y=0}^{d-1} |y\rangle_Y \langle y| \otimes Z_B^y \left( \sum_{j=0}^{d-1} \tau_A^j \otimes \frac{1}{d} |\tilde{e}_j\rangle_B \langle \tilde{e}_j| \right) Z_B^{-y}, \quad (68)$$

$$\Omega_{AB} = \rho_A \otimes \frac{I_B}{d}, \quad \Omega_{XY} = \frac{I_{XY}}{d^2}, \quad (69)$$

and

$$\Omega_X = \frac{I_X}{d}, \quad \Omega_Y = \frac{I_Y}{d}. \quad (70)$$

For the Tsallis- $q$  mutual entropies of  $\Omega_{XYAB}$ ,  $\Omega_{XAB}$  and  $\Omega_{YAB}$  in Eqs. (66), (67) and (68), we have

$$I_q(\Omega_{XY:AB}) = \frac{d^{1-q} - 1}{1 - q} + d^{1-q} S_q(\rho_A) - d^{2(1-q)} S_q(\rho_{AB}), \quad (71)$$

$$I_q(\Omega_{X:AB}) = \frac{d^{1-q} - 1}{1 - q} - d^{1-q} S_q(\rho_B) + d^{1-q} \chi_q(\mathcal{E}_0) \quad (72)$$

and

$$I_q(\Omega_{Y:AB}) = (1 - d^{1-q}) \frac{d^{1-q} - 1}{1 - q} + d^{1-q} \chi_q(\mathcal{E}_1), \quad (73)$$

where the detail calculations can be found in Appendix A.

## VI. GENERAL POLYGAMY INEQUALITY OF MULTI-PARTY QUANTUM ENTANGLEMENT IN TERMS OF TSALLIS ENTROPY

In this section, we provide some sufficient condition for the general polygamy inequality of multi-party quantum entanglement in arbitrary dimensions using Tsallis- $q$  entropy. The following theorem shows that the subadditivity of Tsallis- $q$  mutual entropy for ccq states implies the polygamy inequality of three-party quantum entanglement in terms of Tsallis- $q$  entanglement.

**Theorem 1.** *For  $q \geq 1$ , and any three-party pure state  $|\psi\rangle_{ABC}$  of arbitrary dimension, we have*

$$\mathcal{T}_q(|\psi\rangle_{A|BC}) \leq \mathcal{T}_q^a(\rho_{A|B}) + \mathcal{T}_q^a(\rho_{A|C}), \quad (74)$$

conditioned on the subadditivity of Tsallis- $q$  mutual entropy for the ccq state in Eq. (66), that is,

$$I_q(\Omega_{XY:AB}) \geq I_q(\Omega_{X:AB}) + I_q(\Omega_{Y:AB}). \quad (75)$$

We note that TEOA in Eq. (21) reduces to EoA in Eq. (23) for the case that  $q = 1$ , where the general polygamy inequality of multi-party entanglement in terms of EoA was shown as Inequality (36) [13]. Thus we show the theorem for  $q > 1$ . We also assume that, without loss of generality,  $|\psi\rangle_{ABC}$  is a three-qudit state, that is,  $|\psi\rangle_{ABC} \in (\mathbb{C}^d)^{\otimes 3}$ , otherwise, we can always consider an imbedded image of  $|\psi\rangle_{ABC}$  into a higher dimensional quantum system having the same dimensions of subsystems.

*Proof.* For the reduced density matrices  $\rho_{AB} = \text{tr}_C |\psi\rangle_{ABC} \langle \psi|$  of  $|\psi\rangle_{ABC}$  on subsystem  $AB$ , let us consider the ccq state in Eq. (66). From Eqs. (72), (73) and (71), we can rewrite Inequality (75) as

$$\chi_q(\mathcal{E}_0) + \chi_q(\mathcal{E}_1) \leq S_q(\rho_A) + S_q(\rho_B) - d^{1-q} S_q(\rho_{AB}) + \frac{(d^{1-q} - 1)^2}{d^{1-q}(1 - q)}. \quad (76)$$

Because  $\chi_q(\mathcal{E}_0)$  and  $\chi_q(\mathcal{E}_1)$  of Eqs. (64) and (65) can be obtained, respectively, from  $\rho_{AB}$  by rank-1 measurements  $\{|e_i\rangle_B \langle e_i|\}_i$  and  $\{|\tilde{e}_j\rangle_B \langle \tilde{e}_j|\}_j$  of subsystem  $B$ , the rank-1 measurement

$$\mathbf{Q}_B := \left\{ \frac{|e_i\rangle_B \langle e_i|}{2}, \frac{|\tilde{e}_j\rangle_B \langle \tilde{e}_j|}{2} \right\}_{i,j}, \quad (77)$$

of subsystem  $B$  provides an upperbound of  $q$ -UE in Eq. (44) as

$$\mathbf{u}E_q^{\leftarrow}(\rho_{AB}) \leq \frac{\chi_q(\mathcal{E}_0) + \chi_q(\mathcal{E}_1)}{2}. \quad (78)$$

Thus, together with Inequality (76), we have

$$\mathbf{u}E_q^{\leftarrow}(\rho_{AB}) \leq \frac{1}{2}[S_q(\rho_A) + S_q(\rho_B) - d^{1-q}S_q(\rho_{AB}) + \frac{(d^{1-q}-1)^2}{d^{1-q}(1-q)}]. \quad (79)$$

Moreover, we also analogously have

$$\mathbf{u}E_q^{\leftarrow}(\rho_{AC}) \leq \frac{1}{2}[S_q(\rho_A) + S_q(\rho_C) - d^{1-q}S_q(\rho_{AC}) + \frac{(d^{1-q}-1)^2}{d^{1-q}(1-q)}], \quad (80)$$

for the reduced density matrix  $\rho_{AC} = \text{tr}_B|\psi\rangle_{ABC}\langle\psi|$  on subsystem  $AC$ .

As  $S_q(\rho_{AB}) = S_q(\rho_C)$  and  $S_q(\rho_{AC}) = S_q(\rho_B)$  for the three-party pure state  $|\psi\rangle_{ABC}$ , Inequalities (51) and (52) together with Inequalities (79) and (80) lead us to

$$\begin{aligned} \mathcal{T}_q(\rho_{A|B}) + \mathcal{T}_q(\rho_{A|C}) &\geq 2S_q(\rho_A) \\ &\quad - \mathbf{u}E_q^{\leftarrow}(\rho_{AB}) - \mathbf{u}E_q^{\leftarrow}(\rho_{AC}) \\ &\geq S_q(\rho_A) + \frac{\Xi_B + \Xi_C}{2} \end{aligned} \quad (81)$$

where

$$\Xi_B = \frac{d^{q-1}-1}{d^{q-1}} \left[ \frac{d^{q-1}-1}{q-1} - S_q(\rho_B) \right] \quad (82)$$

and

$$\Xi_C = \frac{d^{q-1}-1}{d^{q-1}} \left[ \frac{d^{q-1}-1}{q-1} - S_q(\rho_C) \right]. \quad (83)$$

For  $q > 1$ , the factor  $\frac{d^{q-1}-1}{d^{q-1}}$  in Eqs. (82) and (83) is nonnegative. Moreover, due to the fact that Tsallis- $q$  entropy attains its maximum value for the maximally mixed state  $\frac{I_B}{d}$ , we have

$$\begin{aligned} S_q(\rho_B) &\leq S_q\left(\frac{I_B}{d}\right) = \frac{1-d^{1-q}}{q-1} \\ &= \frac{d^{q-1}-1}{d^{q-1}(q-1)} \\ &\leq \frac{d^{q-1}-1}{q-1}, \end{aligned} \quad (84)$$

for  $q > 1$ . Similarly, we have

$$S_q(\rho_C) \leq \frac{d^{q-1}-1}{q-1}, \quad (85)$$

and thus

$$\Xi_B \geq 0, \quad \Xi_C \geq 0 \quad (86)$$

for  $q > 1$ .

Inequality (81) together Inequalities (86), we have

$$S_q(\rho_A) \leq \mathcal{T}_q(\rho_{A|B}) + \mathcal{T}_q(\rho_{A|C}), \quad (87)$$

which recovers Inequality (74) because  $\mathcal{T}_q(|\psi\rangle_{A|BC}) = S_q(\rho_A)$  for three-party pure state  $|\psi\rangle_{ABC}$ .  $\square$

We note that, for  $q = 1$ , Tsallis- $q$  mutual entropy is reduced to the quantum mutual information, which is subadditive for ccq-states (Appendix B). Thus Theorem 1 guarantees the general polygamy inequality of TEoA without the subadditivity condition (75) for  $q = 1$ . This also recovers the results in [13].

Now, we generalize the polygamy inequality of three-party quantum entanglement in Theorem 1 into an arbitrary multi-party quantum systems.

**Theorem 2.** *For  $q \geq 1$ , the general polygamy inequality multi-party quantum entanglement,*

$$\mathcal{T}_q^a(\rho_{A_1|A_2\cdots A_n}) \leq \mathcal{T}_q^a(\rho_{A_1|A_2}) + \cdots + \mathcal{T}_q^a(\rho_{A_1|A_n}) \quad (88)$$

*holds for any multi-party quantum state  $\rho_{A_1A_2\cdots A_n}$  of arbitrary dimension, conditioned on the subadditivity of Tsallis- $q$  mutual entropy for the ccq state in Eq. (66).*

*Proof.* We first prove the theorem for a three-party mixed state  $\rho_{ABC}$ , and inductively show the validity of the theorem for an arbitrary  $n$ -party quantum state  $\rho_{A_1A_2\cdots A_n}$ .

For a three-party mixed state  $\rho_{ABC}$ , let us consider an optimal decomposition of  $\rho_{ABC}$  for TEoA with respect to the bipartition between  $A$  and  $BC$ , that is,

$$\rho_{ABC} = \sum_i p_i |\psi_i\rangle_{ABC}\langle\psi_i|, \quad (89)$$

with

$$\mathcal{T}_q^a(\rho_{A|BC}) = \sum_i p_i \mathcal{T}_q^a(|\psi_i\rangle_{A|BC}). \quad (90)$$

From Theorem 1, each  $|\psi_i\rangle_{ABC}$  in Eq. (90) satisfies the polygamy inequality,

$$\mathcal{T}_q(|\psi_i\rangle_{A|BC}) \leq \mathcal{T}_q^a(\rho_{A|B}^i) + \mathcal{T}_q^a(\rho_{A|C}^i) \quad (91)$$

with  $\rho_{AB}^i = \text{tr}_C|\psi_i\rangle_{ABC}\langle\psi_i|$  and  $\rho_{AC}^i = \text{tr}_B|\psi_i\rangle_{ABC}\langle\psi_i|$ , therefore, together with Eq. (90), we have

$$\begin{aligned} \mathcal{T}_q^a(\rho_{A|BC}) &\leq \sum_i p_i \mathcal{T}_q^a(\rho_{A|B}^i) + \sum_i p_i \mathcal{T}_q^a(\rho_{A|C}^i) \\ &\leq \mathcal{T}_q^a(\rho_{A|B}) + \mathcal{T}_q^a(\rho_{A|C}) \end{aligned} \quad (92)$$

where the second inequality is from the definition of TEoA.

Now let us assume Inequality (92) is true for and  $(n-1)$ -party quantum state, and consider an  $n$ -party quantum state  $\rho_{A_1A_2\cdots A_n}$ . By considering  $\rho_{A_1A_2\cdots A_n}$  as a three-party state with respect to the partition  $A_1$ ,  $A_2$  and  $A_3\cdots A_n$ , Inequality (92) leads us to

$$\mathcal{T}_q^a(\rho_{A_1|A_2\cdots A_n}) \leq \mathcal{T}_q^a(\rho_{A_1|A_2}) + \mathcal{T}_q^a(\rho_{A_1|A_3\cdots A_n}), \quad (93)$$

where  $\rho_{A_1A_2} = \text{tr}_{A_3\cdots A_n}\rho_{A_1A_2\cdots A_n}$ ,  $\rho_{A_1A_3\cdots A_n} = \text{tr}_{A_2}\rho_{A_1A_2\cdots A_n}$ , and  $\mathcal{T}_q^a(\rho_{A_1|A_3\cdots A_n})$  is TEoA of  $\rho_{A_1A_3\cdots A_n}$  with respect to the bipartition between  $A_1$  and  $A_3\cdots A_n$ .



Because  $\rho_{A_1 A_3 \dots A_n}$  in Inequality (93) is a  $(n-1)$ -party quantum state, the induction hypothesis assures that

$$\mathcal{T}_q^a(\rho_{A_1|A_3 \dots A_n}) \leq \mathcal{T}_q^a(\rho_{A_1|A_3}) + \dots + \mathcal{T}_q^a(\rho_{A_1|A_n}). \quad (94)$$

Thus Inequalities (93) and (94) imply the polygamy inequality of multi-party entanglement in terms of TEOA in (88).  $\square$

Due to the relation between TEOA and EoA in Eq. (22), Tsalli- $q$  polygamy inequality in (88) is reduced to EoA-based polygamy inequality in (36) for  $q = 1$ . As the quantum mutual information is subadditive for ccq-states (Appendix B), Theorem 2 is true without the subadditivity condition for  $q = 1$ , which encapsulates the results in [14].

## VII. CONCLUSION

We have established a unified view to polygamy inequalities of multi-party quantum entanglement in arbitrary dimensions using Tsallis- $q$  entropy. We have provided a one-parameter class of polygamy inequalities in multi-party quantum systems in terms of TEOA, which provides an interpolation among various polygamy inequalities of multi-party quantum entanglement.

We have further provided one-parameter generalizations of Holevo quantity, UE and quantum mutual in-

formation. By investigating the properties of the generalized quantum correlations related with four-party ccq-states, we have provided a sufficient condition, on which the Tsallis- $q$  polygamy inequality holds in multi-party quantum systems of arbitrary dimensions. We have also shown that the sufficient condition is guaranteed for  $q = 1$ , which is the case that Tsallis- $q$  polygamy inequality is reduced to the general polygamy inequality based on EoA. Thus our results encapsulate the known results of EoA-based general polygamy inequality in a unified view in terms of Tsallis- $q$  entropy.

Based on the concept of  $q$ -expectation, our results provide one-parameter classes of various quantum correlations as well as their properties, which are useful methods in establishing general polygamy of multi-party entanglement in arbitrary dimensions. Noting the importance of the study on multi-party quantum entanglement, especially in higher-dimensional systems more than qubits, our result can provide a rich reference for future work to understand the nature of multi-party quantum entanglement.

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### Appendix A: Tsallis- $q$ mutual entropy of ccq-states

Here, we provide the detail calculation of the Tsallis- $q$  mutual entropies of the ccq-state  $\Omega_{XAB}$  in Eq. (71) as well as the reduced density matrices  $\Omega_{XAB}$  and  $\Omega_{YAB}$  in Eqs. (72) and (73). Let us first consider  $\Omega_{XAB}$ . From the definition of Tsallis- $q$  mutual entropy in Eq. (53), we have

$$I_q(\Omega_{X:AB}) = S_q(\Omega_X) + S_q(\Omega_{AB}) - S_q(\Omega_{XAB}), \quad (\text{A1})$$

where Eq. (70) implies that

$$S_q(\Omega_X) = S_q\left(\frac{I_X}{d}\right) = \frac{d^{1-q} - 1}{1 - q}. \quad (\text{A2})$$

From Eq. (69), we also have

$$\begin{aligned} S_q(\Omega_{AB}) &= S_q\left(\rho_A \otimes \frac{I_B}{d}\right) \\ &= S_q(\rho_A) + S_q\left(\frac{I_B}{d}\right) + (1 - q)S_q(\rho_A)S_q\left(\frac{I_B}{d}\right) \\ &= \frac{d^{1-q} - 1}{1 - q} + d^{1-q}S_q(\rho_A) \end{aligned} \quad (\text{A3})$$

where the second equality is due to the pseudoadditivity of Tsallis- $q$  entropy in Eq. (8). For  $S_q(\Omega_{XAB})$ , the joint entropy theorem in Lemma 1 implies that

$$\begin{aligned} S_q(\Omega_{XAB}) &= H_q(\mathbf{I}_d) + \sum_{x=0}^{d-1} \frac{1}{d^q} S_q\left(\sum_i \sigma_A^i \otimes \lambda_i |e_i\rangle_B \langle e_i|\right) \\ &= H_q(\mathbf{I}_d) + d^{1-q} \left[ H_q(\mathbf{\Lambda}) + \sum_i \lambda_i^q S_q(\sigma_A^i) \right] \end{aligned} \quad (\text{A4})$$

where  $\mathbf{I}_d = \{1/d, \dots, 1/d\}$  is the uniform probability distribution and  $\mathbf{\Lambda} = \{\lambda_i\}_i$  is the spectrum of  $\rho_B$ .

Due to the relation

$$H_q(\mathbf{I}_d) = \frac{d^{1-q} - 1}{1 - q} = S_q\left(\frac{I_B}{d}\right) \quad (\text{A5})$$

and

$$H_q(\mathbf{\Lambda}) = S_q(\rho_B), \quad (\text{A6})$$

Eqs. (A2), (A3) and (A4) lead us to

$$\begin{aligned} I_q(\Omega_{X:AB}) &= \frac{d^{1-q} - 1}{1 - q} + d^{1-q}S_q(\rho_A) - d^{1-q} \left[ H_q(\mathbf{\Lambda}) + \sum_i \lambda_i^q S_q(\sigma_A^i) \right] \\ &= \frac{d^{1-q} - 1}{1 - q} - d^{1-q}S_q(\rho_B) + d^{1-q}\chi_q(\mathcal{E}_0), \end{aligned} \quad (\text{A7})$$

where  $\chi_q(\mathcal{E}_0)$  is the Tsallis- $q$  difference of the induced ensemble  $\mathcal{E}_0$  in Eq. (64).

For the Tsallis- $q$  mutual entropy of  $\Omega_{YAB}$ , we have

$$\begin{aligned} I_q(\Omega_{Y:AB}) &= S_q(\Omega_Y) + S_q(\Omega_{AB}) - S_q(\Omega_{YAB}) \\ &= 2\frac{d^{1-q} - 1}{1 - q} + d^{1-q}S_q(\rho_A) - S_q(\Omega_{YAB}). \end{aligned} \quad (\text{A8})$$

Because

$$\begin{aligned} S_q(\Omega_{YAB}) &= H_q(\mathbf{I}_d) + \sum_{y=0}^{d-1} \frac{1}{d^q} S_q\left(\sum_j \tau_A^i \otimes \frac{1}{d} |\tilde{e}_j\rangle_B \langle \tilde{e}_j|\right) \\ &= (1 + d^{1-q})H_q(\mathbf{I}_d) + d^{1-q} \sum_j \frac{1}{d^q} S_q(\tau_A^j), \end{aligned} \quad (\text{A9})$$

where the second equality is due to the joint entropy theorem in Lemma 1, Eqs. (A8) and (A9) lead us to

$$I_q(\Omega_{Y:AB}) = (1 - d^{1-q}) \frac{d^{1-q} - 1}{1 - q} + d^{1-q} \chi_q(\mathcal{E}_1), \quad (\text{A10})$$

where  $\chi_q(\mathcal{E}_1)$  is the Tsallis- $q$  difference of the induced ensemble  $\mathcal{E}_1$  in Eq. (65).

For the Tsallis- $q$  mutual entropy of  $\Omega_{XYAB}$ , we have

$$I_q(\Omega_{XY:AB}) = S_q(\Omega_{XY}) + S_q(\Omega_{AB}) - S_q(\Omega_{XYAB}), \quad (\text{A11})$$

where Eqs. (69) imply that

$$S_q(\Omega_{XY}) = S_q\left(\frac{I_{XY}}{d^2}\right) = H_q(\mathbf{I}_{\mathbf{d}^2}), \quad (\text{A12})$$

for the uniform probability distribution  $\mathbf{I}_{\mathbf{d}^2} = \{1/d^2, \dots, 1/d^2\}$ . Moreover, from the joint entropy theorem in Lemma 1, we have

$$S_q(\Omega_{XYAB}) = H_q(\mathbf{I}_{\mathbf{d}^2}) + \sum_{x,y} \frac{1}{d^{2q}} S_q(\rho_{AB}), \quad (\text{A13})$$

therefore Eq. (A11) together with Eqs. (A3), (A12) and (A13) lead us to

$$I_q(\Omega_{XY:AB}) = \frac{d^{1-q} - 1}{1 - q} + d^{1-q} S_q(\rho_A) - d^{2(1-q)} S_q(\rho_{AB}). \quad (\text{A14})$$

## Appendix B: Subadditivity of quantum mutual information for ccq-states

Here we provide a detail proof that the quantum mutual information in Eq. (54) is subadditive for general ccq-states of the form

$$\Gamma_{XYAB} = \frac{1}{d^2} \sum_{x,y=0}^{d-1} |x\rangle_X \langle x| \otimes |y\rangle_Y \langle y| \otimes \sigma_{AB}^{xy}, \quad (\text{B1})$$

which has the ccq-state in Eq. (66) as a special case. Then the subadditivity of quantum mutual information for  $\Gamma_{XYAB}$  in Eq. (B1) is equivalent to the nonnegativity

$$I(\Gamma_{XY:AB}) - I(\Gamma_{X:AB}) - I(\Gamma_{Y:AB}) \geq 0. \quad (\text{B2})$$

Let us first consider the mutual information

$$I(\Gamma_{XY:AB}) = S(\Gamma_{XY}) + S(\Gamma_{AB}) - S(\Gamma_{XYAB}). \quad (\text{B3})$$

Due to the joint entropy theorem in Eq. (13), the von Neumann entropy of  $\Gamma_{XYAB}$  is

$$S(\Gamma_{XYAB}) = H(\mathbf{I}_{\mathbf{d}^2}) + \sum_{x,y} \frac{1}{d^2} S(\sigma^{xy}) = 2 \log d + \sum_{x,y} \frac{1}{d^2} S(\sigma^{xy}). \quad (\text{B4})$$

for the uniform probability distribution  $\mathbf{I}_{\mathbf{d}^2} = \{1/d^2, \dots, 1/d^2\}$ .

Because the reduced density matrices

$$\Gamma_{XY} = \frac{1}{d^2} \sum_{x,y=0}^{d-1} |x\rangle_X \langle x| \otimes |y\rangle_Y \langle y| \quad (\text{B5})$$

is a  $d^2$ -dimensional maximally mixed state, its von Neumann entropy is

$$S(\Gamma_{XY}) = S\left(\frac{I_{XY}}{d^2}\right) = 2 \log d. \quad (\text{B6})$$

Thus, together with the reduced density matrix

$$\Gamma_{AB} = \frac{1}{d^2} \sum_{x,y=0}^{d-1} \sigma_{AB}^{xy}, \quad (\text{B7})$$

Eqs. (B4), (B6) imply

$$\begin{aligned} I(\Gamma_{XY:AB}) &= S(\Gamma_{XY}) + S(\Gamma_{AB}) - S(\Gamma_{XYAB}) \\ &= S\left(\frac{1}{d^2} \sum_{x,y=0}^{d-1} \sigma_{AB}^{xy}\right) - \sum_{x,y} \frac{1}{d^2} S(\sigma_{AB}^{xy}). \end{aligned} \quad (\text{B8})$$

Similarly, for the reduced density matrices

$$\Gamma_{XAB} = \frac{1}{d} \sum_{x=0}^{d-1} \left( |x\rangle_X \langle x| \otimes \sum_{y=0}^{d-1} \sigma_{AB}^{xy} \right) \quad (\text{B9})$$

and

$$\Gamma_{YAB} = \frac{1}{d} \sum_{y=0}^{d-1} \left( |y\rangle_Y \langle y| \otimes \sum_{x=0}^{d-1} \sigma_{AB}^{xy} \right), \quad (\text{B10})$$

we have

$$I(\Gamma_{X:AB}) = S\left(\frac{1}{d^2} \sum_{x,y=0}^{d-1} \sigma_{AB}^{xy}\right) - \sum_{x=0}^{d-1} \frac{1}{d} S\left(\sum_{y=0}^{d-1} \sigma_{AB}^{xy}\right) \quad (\text{B11})$$

and

$$I(\Gamma_{Y:AB}) = S\left(\frac{1}{d^2} \sum_{x,y=0}^{d-1} \sigma_{AB}^{xy}\right) - \sum_{y=0}^{d-1} \frac{1}{d} S\left(\sum_{x=0}^{d-1} \sigma_{AB}^{xy}\right). \quad (\text{B12})$$

From Eqs. (B8), (B11) and (B12), the nonnegativity in (B2) can be rephrased as

$$\sum_{y=0}^{d-1} \frac{1}{d} \left[ S\left(\sum_{x=0}^{d-1} \frac{1}{d} \sigma_{AB}^{xy}\right) - \sum_{x=0}^{d-1} \frac{1}{d} S(\sigma_{AB}^{xy}) \right] \geq S\left(\sum_{x,y=0}^{d-1} \frac{1}{d^2} \sigma_{AB}^{xy}\right) - \sum_{x=0}^{d-1} \frac{1}{d} S\left(\sum_{y=0}^{d-1} \frac{1}{d} \sigma_{AB}^{xy}\right). \quad (\text{B13})$$

Now, let us denote

$$\rho = \sum_{x,y=0}^{d-1} \frac{1}{d^2} \sigma_{AB}^{xy} \quad (\text{B14})$$

and consider a probability ensemble of  $\rho$

$$\mathcal{E}_x = \left\{ \frac{1}{d}, \rho^x \right\}, \quad \rho^x = \sum_{y=0}^{d-1} \frac{1}{d} \sigma_{AB}^{xy} \quad (\text{B15})$$

for each  $x$ . Then the right-hand side of Inequality (B13) is the Holevo quantity of  $\rho$  with respect to the ensemble  $\mathcal{E}_x$ ,

$$\chi(\mathcal{E}_x) = S(\rho) - \sum_{x=0}^{d-1} \frac{1}{d} S(\rho^x), \quad (\text{B16})$$

which also has an alternative representation

$$\chi(\mathcal{E}_x) = \sum_{x=0}^{d-1} \frac{1}{d} S(\rho^x \| \rho) \quad (\text{B17})$$

in terms of the quantum relative entropy

$$S(\rho\|\sigma) = \text{tr}\rho \log \rho - \text{tr}\rho \log \sigma. \quad (\text{B18})$$

By denoting

$$\rho^y = \sum_{x=0}^{d-1} \frac{1}{d} \sigma_{AB}^{xy} \quad (\text{B19})$$

and considering a probability ensemble of  $\rho^y$

$$\mathcal{E}_y = \left\{ \frac{1}{d}, \sigma_{AB}^{xy} \right\} \quad (\text{B20})$$

for each  $y$ , a similar argument enables us to rephrase the left-hand side of Inequality (B13) as

$$\sum_{y=0}^{d-1} \frac{1}{d} \left[ \sum_{x=0}^{d-1} \frac{1}{d} S(\sigma_{AB}^{xy} \|\rho^y) \right]. \quad (\text{B21})$$

From Inequality (B13) together with Eqs. (B17) and (B21), the nonnegativity in (B2) is now equivalent to

$$\sum_{x,y=0}^{d-1} \frac{1}{d^2} S(\sigma_{AB}^{xy} \|\rho^y) \geq \sum_{x=0}^{d-1} \frac{1}{d} S(\rho^x \|\rho) \quad (\text{B22})$$

which is always true due to the joint convexity of quantum relative entropy

$$\sum_i p_i S(\rho_i \|\sigma_i) \geq S\left(\sum_i p_i \rho_i \parallel \sum_i p_i \sigma_i\right), \quad (\text{B23})$$

for quantum states  $\rho_i$ 's,  $\sigma_i$ 's and a probability distribution  $\{p_i\}$ .

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